EXAMINATION GENERAL RELATIVITY

Thursday November 4, 2021

Indicate at the first page clearly your name and student number.

Question 1: A Two-dimensional Spacetime with a Horizon (50 pt)

Consider the two-dimensional spacetime spanned by the coordinates (v, x) with the line element

$$ds^2 = -x dv^2 + 2 dv dx \,. \tag{1}$$

(1.1) Give the expressions for the non-zero metric components $g_{\alpha\beta}(x)$. (10 pt)

(1.2) Derive the expressions for the non-zero inverse metric components $g^{\alpha\beta}(x)$. (10 pt)

(1.3) Calculate the curves of lightrays and draw them in a (v, x) spacetime diagram. Argue which of the light rays, with increasing v, are left-moving and which are right-moving. Hint: you may consider v as an evolution parameter and assume that one class of lightrays you will find, i.e. the light-rays with $v =$ constant, are all left-moving. (10 pt)

(1.4) Take $v = 1$ and show how the lightcones change for different values of x. Use these lightcones to draw (without deriving a formula) the timelike curve of a free massive particle that goes through the point $(v, x) = (1, 0)$. (10 pt)

(1.5) Argue that the surface $x = 0$ acts as a horizon in the sense that a free massive particle can move from positive x to negative x but not the other way around. (10 pt)

Question 2: A Three-dimensional Geometry (50 pt)

Consider a three-dimensional spacetime with coordinates $x^\alpha = (t, r, \phi)$ and the following line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 . \tag{2}$$

(2.1) The action for the variational principle for geodesics $x^\alpha(\sigma)$ in this spacetime is given by

$$S = \int d\tau = \int d\sigma \left[-\frac{ds^2}{d\sigma^2} \right]^{1/2} \tag{3}$$

Show that the corresponding Lagrangian is given by the following two expressions:

$$L(\dot{t}, \dot{r}, \dot{\phi}, r) = \left[\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 \right]^{1/2} \tag{4}$$

and

$$L(\dot{t}, \dot{r}, \dot{\phi}, r) = \frac{d\tau}{d\sigma} \tag{5}$$

with $\dot{x}^\alpha = dx^\alpha / d\sigma$. (10 pt)

(2.2) Using the explicit form of the line element (2), discuss the symmetries of the Lagrangian (4). What is the physical interpretation of the corresponding conserved quantities? (10 pt)

(2.3) Vary the Lagrangian (4) with respect to the t coordinate and show that the corresponding Euler-Lagrange equation of motion is given by (10 pt)

$$\frac{d}{d\tau} \left[\left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \right] = 0 . \tag{6}$$

(2.4) Using this equation of motion read of the expressions for the non-zero Christoffel symbols $\Gamma^t_{\alpha\beta}$ by comparing with the general expression of the geodesic equation (10 pt)

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0 . \tag{7}$$

(2.5) Calculate the expressions for the non-zero Christoffel symbols $\Gamma^t_{\alpha\beta}$ also 'by hand' using the formula

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right) \tag{8}$$

and compare your result with the expressions calculated in (1.4). (10 pt)

2.4

Question 3: The relativistic Kepler's Law (50 pt)

A spaceship is moving without power in a circular orbit in the equatorial plane (with $\theta = \pi/2$) about a black hole of mass M with angular velocity

$$\Omega \equiv \frac{d\phi}{dt} . \tag{9}$$

The exterior geometry is the Schwarzschild geometry with the following line element in the coordinates (t, r, θ, ϕ):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{10}$$

The radius of the orbit is given by $r = R$.

(3.1) Show that the angular velocity Ω is given by

$$\Omega = \frac{1}{R^2}\left(1 - \frac{2M}{R}\right)\left(\frac{\ell}{e}\right) , \tag{11}$$

with the conserved quantities e and ℓ (for $r = R$) given by (10 pt)

$$e = \left(1 - \frac{2M}{R}\right)\frac{dt}{d\tau} , \qquad \ell = R^2\frac{d\phi}{d\tau} . \tag{12}$$

(3.2) The radius of the circular orbit is given by the following minimum of the effective potential:

$$R = \frac{\ell^2}{2M}\left[1 + \sqrt{1 - 12(M/\ell)^2}\right] . \tag{13}$$

Furthermore, for a stable circular orbit the energy is equal to the effective potential which implies

$$e^2 = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) . \tag{14}$$

Use these equations to show that

$$\frac{\ell}{e} = (MR)^{1/2}\left(1 - \frac{2M}{R}\right)^{-1} . \tag{15}$$

Hint: first use eq. (13) to solve for ℓ^2 in terms of M and R. (15 pt)

(3.3) Use the previous equations to derive the relativistic Kepler's law (10 pt)

$$\Omega^2 = \frac{M}{R^3} . \tag{16}$$

(3.4) What is the period of the orbit of the spaceship, in terms of R and M, as measured by an observer at infinity? (5 pt)

(3.5) Calculate the angular velocity of the spaceship as measured by a clock in the spaceship. Hint: Use as input the normalization $\mathbf{u} \cdot \mathbf{u} = -1$ of the four-velocity vector. (10 pt)